STAT 510: Homework 02

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Due: Friday, September 11, 11:59 PM

General Directions

This assignment is worth 10 points. For each exercise, you may obtain a score of 0, 0.5, or 1.

- To obtain a score of **1**, your answer must be correct, contain valid supporting work, and be reasonably formatted up to and including boxing your answer when possible.
- A score of **0.5** will be given to solutions which show reasonabe effort, but contain errors. (A score of **1** may be granted to some solutions containing errors if they are extremely minor.)
- A score of **0** will be given to a blank solution or a solution that shows no reasonable progress towards the correct solution. Note that if you do not indicate a page for a problem on Gradescope, it will be considered blank.

Please submit your assignment to Gradescope by the due date listed above. You may submit up to 48 hours late with a two point late penalty. After that, no late work will be accepted.

Any grade disputes must be petitioned through Gradescope within one week of receiving a grade.

Please attempt to submit your work as a single PDF and complete the process of indicating which problem is on which page. You may need to merge together PDF files from various sources and scans. We will keep track of best practice for submitting to Gradescope in this Piazza thread.

Homework assignments are meant to be learning experiences. You may discuss the exercises with other students, but you must write the solutions on your own. Directly sharing or copying any part of a homework solution is an infraction of the University's rules on academic integrity. Any violation will be punished as severely as possible.

For this, and all homework assignments, you may use any computational tools that you wish, such as a statistical computing enviroment or integral solver. The course staff is most familiar with `R`, so we will be able to best support `R` users, but you may use any software that you like.

Practice Exercises

The following exercises from Evans and Rosenthal contain back-of-the-book solutions.

- Section 3.1: 1, 3, 5, 7, 9, 11, 13
- Section 3.2: 1, 3, 5, 7, 9, 11, 13, 15
- Section 3.3: 1, 3, 5, 7, 9, 11, 13, 15
- Section 3.4: 1, 3, 5, 7
- Section 3.5: 1, 3, 7, 9, 11
- Section 3.7: 1, 3, 5, 7, 9

This is an unusually large number of practice problems. In the future there may be far fewer. You do not *need* to do each of these exercises. However, we expect that you are able to complete each of these exercises based on previous coursework. While exam problems will generally be most similar to graded exercises, some exercises similar to practice exercises in difficulty may appear on the exam to stabalize the variance of scores.

Graded Exercises

Exercise 1 (Expectation of a Maximum)

(**LW** 3.3) Let $X_1, \ldots, X_n \sim \text{Uniform}(0,1)$. Define $Y_n = \max\{X_1, \ldots, X_n\}$. Find $\mathbb{E}[Y_n]$.

Exercise 2 (A Random Walk)

(**LW** 3.4) A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is $1-p$ that the particle will jump one unit to the right. Let X_n be the position of the particle after n jumps. Find $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Exercise 3 (A Lazy Statistican Does Algebra)

(**LW** 3.10) Let $X \sim \text{Normal}(0,1)$. Define $Y = e^X$. Find $\mathbb{E}[Y]$ and $\mathbb{V}[Y]$. Your answer should be a function of e, not a decimal representation.

Exercise 4 (A Simple Hierarchical Model)

(**LW** 3.13) Suppose we generate a random variable X in the following way. First we flip a fair coin. If the coin is heads, take X to have a $\text{Uniform}(0,1)$ distribution. If the coin is tails, take X to have a $\text{Uniform}(3,4)$ distribution. Find the mean and standard deviation of X.

Exercise 5 (Variance and Covariance)

(**LW** 3.15) Let

$$f_{X,Y}(x,y) = \frac{1}{3}(x+y), \quad 0 < x < 1,\ 0 < y < 2$$

Find $\mathbb{V}[2X - 3Y + 8]$.

Exercise 6 (Checking Independence)

(**LW** 3.22) Let $X \sim \text{Uniform}(0,1)$. Let $0 < a < b < 1$. Let

$$Y = \begin{cases} 1 & 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

and let

$$Z = \begin{cases} 1 & a < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Are Y and Z independent? (Why or why not?) Find $\mathbb{E}[Y \mid Z]$.

Exercise 7 (Using Moment Generating Functions)

(**LW** 3.24) Let $X_1, \ldots, X_n \sim \text{Exp}(\beta)$. Find the moment generating function of X_i and use this to show that

$$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta).$$

Exercise 8 (The Classic Setup)

(**LW** 3.8) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2$.

Prove that,

$$\mathbb{E}[\bar{X}_n] = \mu, \quad \mathbb{V}[\bar{X}_n] = \frac{\sigma^2}{n}, \quad \text{and} \quad \mathbb{E}[S_n^2] = \sigma^2.$$

Exercise 9 (Unintuitive Intuitions)

(Based on **LW** 3.9) Let $X_1, \ldots, X_n \sim \text{Normal}(0, 1)$ and $Y_1, \ldots, Y_n \sim \text{Cauchy}$. (In particular a Cauchy distribution with a location parameter of 0 and a scale parameter of 1.) Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

Generate a (single) random sample of size $n = 10,000$ for both distributions and plot $\bar{x}_n$ and $\bar{y}_n$ versus n for $n = 1, \ldots, 10,000$ on a single plot. (Use a different color for each distribution.) Repeat this process three times. Display the plots in a 1×3 grid. Explain why there is such a difference between the two distributions.

Exercise 10 (Simulating a Stock Market)

(Based on **LW** 2.11) Let $Y_1, Y_2, \ldots$ be independent random variables such that

$$P(Y_i = -1) = P(Y_i = 1) = 0.5.$$

Define

$$X_n = \sum_{i=1}^{n} Y_i.$$

Think of $Y_i = 1$ as "the stock price increased by one dollar," $Y_i = -1$ as "the stock price decreased by one dollar," and X_n as the value of the stock on day n. Find $\mathbb{E}[X_n]$ and $\mathbb{V}[X_n]$.

Simulate X_n and plot X_n versus n for $n = 10,000$. Repeat this process three times. (So, simulate the price of three stocks for $10,000$ days.) Use a single plot with a different color for each stock. Notice two things. First, it is easy to "see" patterns in the sequence even though it is random. Second, you will find that the three runs look very different even though they were generated the same way. How do the expectations you found explain this observation? (Also, consider repeating this process more times than needed to simulate three stocks for more intuition.)

Note: This is an incredibly simplistic model of a market. Please do not make any decisions in the real world based on this model.

Exercise 11 (BYOQ: Bring Your Own Question)

Submit your own question with a solution! If accepted, you will receive one **buffer point**. To be accepted:

- Your question must be reasonably challenging. It should be at least as challenging as the "average" question on this assignment.
- Is must be **original**. If, based on some quick searching we can find the *exact* question (that is, you just copy-pasted a question), you will **lose** a point instead of receiving a buffer point. If we find that it is simply a derivative of another exercise (for example, just changing a constant) you will not receive a point.

The instructor may create videos solving these, or they may be circulated for additional practice.